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On One-Dimensional Stretching Functions for Finite-Difference Calculations*

MARCEL VINOKUR

Mechanical Engineering Research, University of Santa Clara, Santa Clara, California 95053

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The class of one-dimensional stretching functions used in finite-difference calculations is studied. For solutions containing a highly localized region of rapid variation, simple criteria for a stretching function are derived using a truncation error analysis. These criteria are used to investigate two types of stretching functions. One is an interior stretching function, for which the location and slope of an interior clustering region are specified. The simplest such function satisfying the criteria is found to be one based on the inverse hyperbolic sine. It was first employed by Thomas *et al.* (*AIAA J.* **10** (1972), 887). The other type of function is a two-sided stretching function, for which the arbitrary slopes at the two ends of the one-dimensional interval are specified. The simplest such general function is found to be one based on the inverse tangent. The special case where the slopes were both equal and greater than one was first employed by Roberts. The general two-sided function has many applications in the construction of finite-difference grids. Examples of such applications are found in the listed references.

I. INTRODUCTION

Finite-difference calculations of fluid flow problems are best carried out using an equispaced grid in a rectangular (or cubic) computational domain, with the flow variables and components of the position vector as dependent variables, and boundary conditions applied at the edges (or faces) of the domain. In order to minimize the number of grid points required for a given accuracy, one seeks boundary-fitted coordinate transformations that cluster points in regions where the dependent variables undergo rapid variation. These regions may be a result of body geometry (very large curvatures or corners), compressibility (entropy layers, shock waves, and contact discontinuities), and viscosity (boundary layers and shear layers). A complex flow may thus contain a variety of such regions of various length scales, and often of unknown location. An ideal grid would adjust with each time or iteration step to maintain optimum clustering. Such adaptive grid methods, which involve the solution of auxiliary equations, have been developed for one-dimensional problems [1–3]. Their extension to complex multidimensional flows is a difficult problem,

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particularly when the regions requiring clustering do not have simple topological properties required by a finite-difference grid.

There are many practical problems in which the locations and length scales of regions of rapid variation can be estimated a priori (e.g., known geometry, attached boundary layers, simple shock wave configurations). In these cases the clustering can be incorporated in automatic grid generators which solve an elliptic boundary-value problem [4–6]. The distribution of grid points on the boundaries is then normally prescribed algebraically, using one-dimensional stretching functions. (Here, stretching function refers to any transformation involving stretching or clustering.) It is also possible to employ stretching functions to obtain a clustered grid from an unclustered grid by applying clustering to one coordinate family only. For some simple geometries, one can construct entire clustered grids purely algebraically, using only one-dimensional stretching and blending functions.

The simplest class of one-dimensional stretching functions is that involving two parameters. In interior stretching functions, the parameters are the location of a single inflection point, and the slope at that point. In two-sided stretching functions, the slopes at the two ends of the one-dimensional interval are specified. The antisymmetric two-sided stretching function (with the same slope at each end) is of special interest, since the portion from the midpoint to either end defines a one-sided stretching function (with the slope given at one end and zero curvature at the other one). A one-sided stretching function can also be obtained as a special case of the interior stretching function, by locating the inflection point at one end. Since the end where the slope is given has zero curvature in this case, these two one-sided stretching functions are of a different nature.

An interior stretching function based on the inverse hyperbolic sine was employed by Thomas *et al.* [7] in a numerical solution of inviscid supersonic flow over a blunt delta wing with elliptical cross section. The function was used to cluster points on the body at the vertices of very eccentric ellipses. The one-sided version clustered points in the flow near the body surface to resolve the entropy layer caused by the bow shock. No derivation of the stretching function was given, and the clustering parameter appearing in it was not related to the length scales of regions of rapid variation in physical space.

An antisymmetric two-sided stretching function of a logarithmic type was employed by Roberts [8] to study boundary layer flows. The heuristic derivation of the function avoided consideration of the truncation errors associated with finite-difference approximations. While this function has been used successfully in many flow calculations, there is a need for a general two-sided function which allows arbitrary stretching or clustering to be specified independently at each end. An application would be problems in which the appropriate length scales requiring clustering are significantly different at the two ends. Another application is the distribution of grid points on a curve which is defined piecewise, where continuity of grid spacing is desired at the ends of the piecewise segments. A third application is the use of such a function as a blending or interpolating function to construct two- and three-dimensional grids using one-dimensional stretching functions and shearing transformation. The construction of a single, well-ordered grid for wing body flows by Vinokur *et al.* [9] is an example of these applications.

The present work has two objectives. One is to obtain simple, rational criteria for one-dimensional stretching functions, by considering the truncation errors inherent in finite-difference approximations. The functions introduced by Thomas *et al.* and Roberts will be found to be the simplest ones satisfying these criteria. The other objective is to derive a simple form of the general two-sided stretching function. Additional details may be found in the appendix of [12].

II. Truncation Error Analysis for One-Dimensional Stretching Functions

An exact analysis of the truncation error inherent in a finite-difference calculation would require knowledge of the equation being solved and the finite-difference approximation that is used. Here we are concerned with the special situation where the solution contains a highly localized region of rapid variation with respect to some coordinate, and we seek approximate criteria for a stretching function that will be independent of the equation or difference algorithm. The quantities that are approximated are in general nonlinear functions of the unknowns and their spatial derivatives. The error analysis will be performed in terms of the fractional truncation errors for the spatial derivatives.

Let the vector function $\mathbf{r}(\bar{t})$ describe a $\bar{\xi}$-coordinate curve, where $\bar{t}$ is any parameter that varies smoothly with arc length. If A and B denote the ends of the curve, we introduce the normalized variables

$$t = (\bar{t} - \bar{t}_A)/(\bar{t}_B - \bar{t}_A) \tag{1a}$$

and

$$\xi = (\bar{\xi} - \bar{\xi}_A)/(\bar{\xi}_B - \bar{\xi}_A), \tag{1b}$$

ranging from 0 to 1. For simplicity, all partial derivatives with respect to ξ or t will be written as total derivatives. Let $\phi(t)$ be any function of the unknowns. Outside of the region where $d\phi/dt = 0$ and $d^2\phi/dt^2 = 0$, we can define a natural length scale of the variation of ϕ with respect to t as

$$L_{\phi t} = \left| \frac{d\phi}{dt} \bigg/ \frac{d^2\phi}{dt^2} \right|. \tag{2}$$

Since the components of $\mathbf{r}$ also enter as dependent variables in the calculation of metrics and Jacobians, we similarly define the natural length scale of the variation of $\mathbf{r}$ with respect to t as

$$L_{rt} = \left| \frac{d\mathbf{r}}{dt} \right| \bigg/ \left| \frac{d^2\mathbf{r}}{dt^2} \right|. \tag{3}$$

Note that if t is proportional to arc length, then L_{rt} is precisely the radius of curvature, normalized by the length of the curve.

Assume first that t is used as the normalized computational variable (i.e., $\xi = t$), with Δt as the uniform grid spacing. Let $\delta\phi/\delta t$ denote the finite-difference approximation to $d\phi/dt$. Using Eq. (2), one can write any first-order accurate approximation as

$$\frac{\delta\phi}{\delta t} = \frac{d\phi}{dt}\left[1 + O(L_{\phi t}^{-1}\,\Delta t)\right]. \tag{4}$$

Similarly, the approximation to $d\mathbf{r}/dt$ can be written as

$$\frac{\delta\mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt}\left[1 + O(L_{rt}^{-1}\,\Delta t)\right]. \tag{5}$$

If $L_{\phi t}^{-1}$ or L_{rt}^{-1} became very large in some localized region, then a prohibitively small Δt would be required to obtain a desired fractional truncation error. Outside of the localized region, the excessive number of grid points would be wasted. The obvious remedy is to seek a new computational variable ξ for which $L_{\phi\xi}^{-1}$ and $L_{r\xi}^{-1}$ remain of $O(1)$, even though $L_{\phi t}^{-1}$ or L_{rt}^{-1} could be locally very large.

With the aid of the identities

$$\frac{d\phi}{d\xi} = \frac{d\phi}{dt}\,\frac{dt}{d\xi} \tag{6}$$

and

$$\frac{d^2\phi}{d\xi^2} = \frac{d^2\phi}{dt^2}\left(\frac{dt}{d\xi}\right)^2 + \frac{d\phi}{dt}\,\frac{d^2t}{d\xi^2}, \tag{7}$$

and definition (2), one can easily show that

$$L_{\phi\xi}^{-1} \leqslant L_{\phi t}^{-1}\,\frac{dt}{d\xi} + L_{t\xi}^{-1}. \tag{8}$$

Similarly, using definition (3), one obtains the inequality

$$L_{r\xi}^{-1} \leqslant L_{rt}^{-1}\,\frac{dt}{d\xi} + L_{t\xi}^{-1}. \tag{9}$$

One criterion for the stretching function $\xi(t)$ is therefore

$$L_{t\xi}^{-1} = O(1). \tag{10}$$

In addition, we require that

$$L_{\phi t}^{-1}\,\frac{dt}{d\xi} = O(1), \tag{11}$$

and

$$L_{rt}^{-1}\frac{dt}{d\xi}=O(1). \tag{12}$$

Consider the case where $L_{\phi t}^{-1}$ is very large in some localized region. It follows from Eq. (11) that

$$\frac{dt}{d\xi}=O(L_{\phi t}) \tag{13}$$

in that region. It is reasonable to assume that the thickness of the localized region, where $L_{\phi t}^{-1}$ could remain large is at most of $O(L_{\phi t})$. We thus require that Eq. (13) be valid over that distance. Furthermore, since $L_{\phi t}^{-1}=O(1)$ outside of the localized region, it follows from Eq. (11) that $dt/d\xi=O(1)$, i.e., that $dt/d\xi$ does not become large anywhere. But these two additional requirements are satisfied if condition (10) is valid. Noting that

$$\frac{d}{dt}\left(\frac{dt}{d\xi}\right)=\frac{d^2t}{d\xi^2}\bigg/\left(\frac{dt}{d\xi}\right), \tag{14}$$

it follows upon integration over a finite interval Δt that

$$\left|\Delta\left(\frac{dt}{d\xi}\right)\right|\leqslant(L_{t\xi}^{-1})_{\max}\,\Delta t. \tag{15}$$

Applying Eq. (15) to the localized region, for which $\Delta t=O(L_{\phi t})$, we find, using Eq. (10), that

$$\left|\Delta\left(\frac{dt}{d\xi}\right)\right|=O(L_{\phi t}). \tag{16}$$

Thus Eq. (13) is satisfied over the entire localized region. Letting $\Delta t=1$ in Eq. (15), we see that $dt/d\xi=O(1)$ is satisfied over the complete range of t. In the event that L_{rt}^{-1} is larger than $L_{\phi t}^{-1}$ in the localized region, then condition (13) is replaced by

$$\frac{dt}{d\xi}=O(L_{rt}). \tag{17}$$

This analysis is easily extended to higher order finite-difference approximations, as well as the treatment of higher derivatives. In order to consider fractional truncation errors due to a second-order accurate approximation to $d\phi/dt$ (and also for regions where $d^2\phi/dt^2=0$), it is appropriate to define a different length scale of the variation of ϕ with respect to t as

$$\bar{L}_{\phi t}=\left|\frac{d\phi}{dt}\bigg/\frac{d^3\phi}{dt^3}\right|^{1/2}. \tag{18}$$

We now require that $\bar{L}_{\phi\xi}^{-1}$ remains of $O(1)$, even though $\bar{L}_{\phi t}^{-1}$ could be locally very large. Using the identity

$$\frac{d^3\phi}{d\xi^3}=\frac{d^3\phi}{dt^3}\left(\frac{dt}{d\xi}\right)^3+3\frac{d^2\phi}{dt^2}\frac{dt}{d\xi}\frac{d^2t}{d\xi^2}+\frac{d\phi}{dt}\frac{d^3t}{d\xi^3}, \tag{19}$$

we obtain the inequality

$$\bar{L}_{\phi\xi}^{-2}\leqslant\left(\bar{L}_{\phi t}^{-1}\frac{dt}{d\xi}\right)^2+3L_{\phi t}^{-1}\frac{dt}{d\xi}L_{t\xi}^{-1}+\bar{L}_{t\xi}^{-2}. \tag{20}$$

In addition to satisfying Eq. (10), the stretching function $\xi(t)$ must satisfy

$$\bar{L}_{t\xi}^{-1}=O(1). \tag{21}$$

Also, if $\bar{L}_{\phi t}^{-1}$ is larger than $L_{\phi t}^{-1}$ in the localized region, condition (13) must be replaced by

$$\frac{dt}{d\xi}=O(\bar{L}_{\phi t}). \tag{22}$$

A first-order finite-difference approximation to $d^2\phi/dt^2$ requires the introduction of the length scale

$$\tilde{L}_{\phi t}=\left|\frac{d^2\phi}{dt^2}\middle/\frac{d^3\phi}{dt^3}\right|. \tag{23}$$

Combining Eqs. (2), (18), and (23), we see that

$$\tilde{L}_{\phi\xi}^{-1}=\bar{L}_{\phi\xi}^{-2}/L_{\phi\xi}^{-1}. \tag{24}$$

Thus conditions (10), (21), and (13) or (22) are sufficient to guarantee that $\tilde{L}_{\phi\xi}^{-1}$ remains of $O(1)$. The length scale $\tilde{L}_{\phi t}$ is also the appropriate one to use at a point where $d\phi/dt=0$. At such a point, using Eqs. (7) and (19), we obtain the relation

$$\tilde{L}_{\phi\xi}^{-1}\leqslant\tilde{L}_{\phi t}^{-1}\frac{dt}{d\xi}+3L_{t\xi}^{-1}. \tag{25}$$

The criteria for $\xi(t)$ is again Eq. (10), with Eq. (13) replaced by

$$\frac{dt}{d\xi}=O(\tilde{L}_{\phi t}) \tag{26}$$

if the localized region of rapid variation occurs around the point $d\phi/dt=0$. Conditions (22) and (26) are to be replaced by analogous ones containing $\bar{L}_{rt}$ and $\tilde{L}_{rt}$, if these are the more significant length scales.

In summary, one first defines length scales appropriate to the difference approximation and the location of the region of rapid variation. The criteria for the stretching function $\xi(t)$ can be stated as:

(1) All the inverse length scales of the variation of t with respect to ξ must be at most of order one throughout the range of t.

(2) The slope $dt/d\xi$ must be of the order of the minimum length scale of the variation of ϕ or $\mathbf{r}$ with respect to t in the localized region of rapid variation.

These criteria will ensure that most of the grid points will be concentrated in the localized region of rapid variation, with a sufficient number of points left in the remainder of the domain. The criteria will be used to investigate the two-parameter stretching functions of the next two sections.

III. A General Two-Sided Stretching Function

In this section we derive a general two-sided stretching function $\xi(t; s_0, s_1)$, where ξ and t are normalized variables defined by Eq. (1), and the parameters s_0 and s_1 are dimensionless slopes at the two ends defined as

$$s_0 = \frac{d\xi}{dt}(0) \tag{27a}$$

and

$$s_1 = \frac{d\xi}{dt}(1). \tag{27b}$$

In order to be useful for constructing finite-difference grids, the function must be monotonic, and satisfy conditions (10) and (21) even if s_0 or s_1 becomes very large. For a general range of applications, it would be desirable for the function to be simple, readily invertible, and to vary continuously over the complete ranges of s_0 and s_1.

An attractive candidate for such a stretching function is a scaled portion of a single, universal function $w(z)$. For a given s_0 and s_1, the stretching function will be obtained by properly scaling the portion of the universal function from corresponding points z_0 and z_1. An additional requirement is that the unnormalized function $\bar{\xi}(\bar{t})$ be independent of the designation of a particular end as A or B. One can easily show that this restricts the universal function to be odd, i.e.,

$$w(-z) = -w(z).$$

The simplest such functions which generate monotonic, readily invertible stretching functions are $\sin z$ and $\tan z$. Their hyperbolic relatives are produced by letting z be complex. The inverse functions are formed by associating z with either t or ξ. One can determine whether either universal function is suitable as a basis for a stretching function by applying conditions (10) and (21) for very large s_0 or s_1. Actually, the most extreme test occurs for the simpler antisymmetric case $s_0 = s_1$, which corresponds to $z_0 = -z_1$, with z being either real or pure imaginary.

An evaluation on the antisymmetric two-sided stretching functions obtained from $\sin z$ and $\tan z$ is carried out for the case $s_0 = s_1 > 1$ in [12]. Only $\tan z$ produces a stretching function satisfying conditions (10) and (21), with the inverse length scales being logarithmically of $O(1)$. The stretching function $\xi(t)$ is a scaled portion of the inverse hyperbolic tangent. Expressing the hyperbolic tangent as a logarithm, we obtain exactly the function derived by Roberts [8]. It turns out that $L_{t\xi}^{-1}$ is a piecewise linear function of t, a property that was used by Roberts to define his function. This suggests a related stretching function, for which $L_{t\xi}^{-1}$ is a piecewise linear function of ξ. The corresponding universal function is the error function $\operatorname{erf} z$. The associated stretching function is also analyzed in [2] and found to satisfy conditions (10) and (21). However, it is not invertible, and has larger maximum inverse length scales than the former stretching function.

On the basis of these considerations, the universal function $w = \tan z$ will be used to obtain a stretching function for arbitrary s_0 and s_1. Introducing the ranges

$$\Delta z = z_1 - z_0 \tag{28}$$

and

$$\Delta w = \tan z_1 - \tan z_0, \tag{29}$$

we can define the normalized variables

$$\xi = (z - z_0)/\Delta z \tag{30}$$

and

$$t = (\tan z - \tan z_0)/\Delta w. \tag{31}$$

The slope of the stretching function is then given by

$$\frac{d\xi}{dt} = \frac{\Delta w}{\Delta z} \cos^2 z. \tag{32}$$

Using the trigonometric identity

$$\tan z_1 - \tan z_0 = \frac{\sin(z_1 - z_0)}{\cos z_1 \cos z_0}, \tag{33}$$

we find for the parameters s_0 and s_1 the relations

$$s_0 = \frac{\sin \Delta z \cos z_0}{\Delta z \cos z_1} \tag{34}$$

and

$$s_1 = \frac{\sin \Delta z \cos z_1}{\Delta z \cos z_0}. \tag{35}$$

This suggests introducing the new parameters

$$B = \sqrt{s_0 s_1} \tag{36}$$

and

$$A = \sqrt{s_0/s_1}. \tag{37}$$

The parameters A and B can then be expressed in terms of z_0 and z_1 as

$$B = \frac{\sin \Delta z}{\Delta z} \tag{38}$$

and

$$A = \frac{\cos z_0}{\cos z_1}. \tag{39}$$

Using the cosine sum identity, we can also write Eq. (39) as

$$A = \cos \Delta z + \tan z_1 \sin \Delta z \tag{40a}$$

and

$$1/A = \cos \Delta z - \tan z_0 \sin \Delta z. \tag{40b}$$

For a given value of B, Δz is obtained by solving Eq. (38). Equation (40) can then be solved to obtain z_0 and Δw for a given value of A. The stretching function obtained from Eqs. (30) and (31) can then be written as

$$t = \frac{\tan(\xi \Delta z + z_0) - \tan z_0}{\Delta w}. \tag{41}$$

While Eq. (41) is a formal expression for the general stretching function, it cannot be used for calculations in its present form. Depending on the value of B, Δz and Δw are either real or pure imaginary. For certain ranges of A and B, z_0 can become complex. Using the tangent sum identity and Eq. (39), we can eliminate z_0 from Eq. (41) and obtain instead

$$t = \frac{\tan \xi \Delta z}{A \sin \Delta z + (1 - \cos \Delta z) \tan \xi \Delta z}. \tag{42}$$

This can be further simplified by noting that $A = 1$ corresponds to the antisymmetric solution which was analyzed in [12]. Let us denote this solution as $u(\xi)$. Setting $A = 1$ in Eq. (42), and using the tangent sum identity, we obtain

$$u = \frac{1}{2} + \frac{\tan[\Delta z(\xi - \frac{1}{2})]}{2 \tan(\Delta z/2)}. \tag{43}$$

In terms of u, Eq. (2) then takes the simple form

$$t = \frac{u}{A + (1 - A)u}, \tag{44}$$

which can be readily inverted as

$$u = \frac{t}{(1/A) + (1 - 1/A)t}. \tag{45}$$

Note that both $u(t)$ and its inverse can be obtained as scaled portions of a rectangular hyperbola. For each function, the slopes at the two ends are A and $1/A$, respectively. Finally, we observe that for calculational purposes, Eqs. (44) and (45) are well behaved in the neighborhood of $A = 1$.

We thus have the remarkable result that one essentially needs to know only the antisymmetric stretching function for the geometric mean of the slopes s_0 and s_1. The square root of the ratio of those slopes determines an additional simple transformation which produces the desired stretching function. Since both Eqs. (43) and (44) are invertible, the resultant stretching function is also invertible. The key trigonometric property making this result possible is that the tangent of a sum is a rational function of the individual tangents. By contrast, the sine of a sum involves the individual sines and cosines, and is not expressible as a rational function of sines alone. An analysis of a stretching function based on $w = \sin z$ has been carried out, but is not presented here. The parameters corresponding to B and A turn out to be the arithmetic mean and difference of the two slopes. A separation into two functions corresponding to Eqs. (43) and (44) is not possible, and one must use the direct form corresponding to Eqs. (41) and 42). For $B > 1$, the inversion involves the solution of a quadratic equation, and the sign of A must be tested in order to choose the appropriate root. It is indeed serendipity that the tangent function dictated by truncation error considerations is also the much simpler one for constructing a general two-sided stretching function.

The calculation of the antisymmetric function depends on the size of B. If $B > 1$, it follows from Eq. (38) that z is imaginary and we obtain the relations

$$B = \frac{\sinh \Delta y}{\Delta y} \tag{46}$$

and

$$u = \frac{1}{2} + \frac{\tanh[\Delta y(\xi - \frac{1}{2})]}{2 \tanh(\Delta y/2)}. \tag{47}$$

The inversion of Eq. (47) yields

$$\xi = \frac{1}{2} + \frac{\tanh^{-1}[(2u-1)\tanh(\Delta y/2)]}{2\Delta y}. \tag{48}$$

Note that the hyperbolic tangent and its inverse can be expressed in terms of exponentials and a logarithm, respectively.

For $B < 1$, Δz is real, and the corresponding results are

$$B = \frac{\sin \Delta x}{\Delta x}, \tag{49}$$

$$u = \frac{1}{2} + \frac{\tan[\Delta x(\xi - \frac{1}{2})]}{2\tan(\Delta x/2)}, \tag{50}$$

and

$$\xi = \frac{1}{2} + \frac{\tan^{-1}[(2u-1)\tan(\Delta x/2)]}{2\Delta x}. \tag{51}$$

When B is very near one, both of the formulations break down, since Δx and Δy approach zero. The appropriate expressions are obtained by expanding Eqs. (49) and (50) in powers of Δx. To first order in $B - 1$, one obtains

$$u \simeq \xi[1 + 2(B-1)(\xi - 0.5)(1-\xi)] \tag{52}$$

and

$$\xi \simeq u[1 - 2(B-1)(u - 0.5)(1-u)]. \tag{53}$$

By scaling half of the functions, one obtains the one-sided stretching functions with s_0 given at $t = 0$ and zero curvature at $t = 1$.

The results are:

$s_0 > 1$

$$s_0 = \frac{\sinh 2\Delta y}{2\,\Delta y}, \tag{54}$$

$$t = 1 + \frac{\tanh[\Delta y(\xi - 1)]}{\tanh \Delta y}, \tag{55}$$

and

$$\xi = 1 + \frac{\tanh^{-1}[(t-1)\tanh \Delta y]}{\Delta y}. \tag{56}$$

$s_0 < 1$

$$s_0 = \frac{\sin 2\Delta x}{2\,\Delta x}, \tag{57}$$

$$t = 1 + \frac{\tan[\Delta x(\xi - 1)]}{\tan \Delta x}, \tag{58}$$

$$\xi = 1 + \frac{\tan^{-1}[(t-1)\tan \Delta x]}{\Delta x}. \tag{59}$$

$s_0 \simeq 1$

$$t = \xi[1 - 0.5(s_0 - 1)(1 - \xi)(2 - \xi)] \tag{60}$$

and

$$\xi = t[1 + 0.5(s_0 - 1)(1 - t)(2 - t)]. \tag{61}$$

The two-sided stretching functions for $B > 1$ and one sided stretching function for $s_0 > 1$ require the inversion of the function

$$y = \sinh x/x. \tag{62}$$

An approximate analytic representation of the inverse function $x = f(y)$, valid over the range of y required by a stretching function, is derived in [12]. The results are as follows:

For $y < 2.7829681$

$$\begin{aligned} x = \sqrt{6\bar{y}}(1 &- 0.15\bar{y} + 0.057321429\bar{y}^2 - 0.024907295\bar{y}^3 \\ &+ 0.0077424461\bar{y}^4 - 0.0010794123\bar{y}^5), \end{aligned} \tag{63}$$

where

$$\bar{y} = y - 1. \tag{64}$$

For $y > 2.7829681$

$$\begin{aligned} x = v + (1 + 1/v)\log(2v) &- 0.02041793 + 0.24902722w \\ &+ 1.9496443w^2 - 2.6294547w^3 + 8.56795911w^4, \end{aligned} \tag{65}$$

where

$$v = \log y \tag{66}$$

and

$$w = 1/y - 0.028527431. \tag{67}$$

The maximum magnitude of the fractional error in y is 0.000267732 for $1 < y < 69.64$. The magnitude of the error reaches 0.00083 at $y = 120.5$. These errors are small enough so that the grids constructed by the resultant stretching functions will exhibit slope discontinuities that are negligible within the accuracy of any practical finite-difference approximation.

The two-sided stretching function for $B < 1$ and one-sided stretching function for $s_0 < 1$ require the inversion of the function

$$y = \sin x/x. \tag{68}$$

An approximate analytic representation of the inverse function is derived in [12]. The results are as follows:

For $y < 0.26938972$

$$x = \pi[1 - y + y^2 - (1 + \pi^2/6)\, y^3 + 6.794732y^4 - 13.205501y^5 + 11.726095y^6]. \tag{69}$$

For $0.26938972 < y < 1$

$$x = \sqrt{6\bar{y}}\,(1 + 0.15\bar{y} + 0.057321429\bar{y}^2 + 0.048774238\bar{y}^3 - 0.053337753\bar{y}^4 + 0.075845134\bar{y}^5), \tag{70}$$

where

$$\bar{y} = 1 - y. \tag{71}$$

The maximum magnitude of the fractional error in y is 0.00019717, which is small enough for numerical applications.

We can now summarize the procedure to obtain a general two-sided stretching function. Let $\bar{\xi}$ be the computational variable along a given coordinate curve, defined by specifying its value at the two ends of the curve. One must choose a physical variable $\bar{t}$ which parametrizes the curve. This could be the arc length or the projected distance on a given straight line (e.g., Cartesian coordinate). The normalized variables ξ and t, ranging from 0 to 1 at the two ends, are then defined by Eqs. (1). Given the dimensionless slopes s_0 and s_1 at the two ends defined by (27), one first calculates the new parameters B and A from (36) and (37). In most applications, one must obtain the values of t corresponding to given (usually equally spaced) values of ξ. This is done by first obtaining the value of the new intermediate variable u, and then calculate t from (44). The form of the function $u(\xi)$ depends on the size of B. If $B > 1$, $u(\xi)$ is given by Eq. (47), where Δy is defined implicitly by (46) and an explicit analytic representation is found from Eqs. (62)–(67). If $B < 1$, $u(\xi)$ is given by Eq. (50), where Δx is defined implicitly by (49), and an explicit analytic representation is found from Eqs. (68)–(71). If B is close to one within some prescribed error bound ($|B - 1| < 0.001$ is suggested), then $u(\xi)$ is given by Eq. (52). For those applications where $\xi(t)$ is desired, one first determines u from Eq. (45), and then calculates ξ from Eqs. (48), (51), or (53), depending on the size of B. One-sided stretching functions, with the slope s_0 defined at $t = 0$, are similarly obtained using Eqs. (54)–(61).

While it is appealing to view the general two-sided stretching function as a distortion of an antisymmetric stretching function via Eqs. (43) and (44), there are advantages in looking at the more basic forms of the solution given by Eqs. (41) and (42). If efficiency in terms of operations count is important, then the optimum form for the case $B > 1$ is derived from Eq. (42), as either

$$t = \frac{\tanh \xi \Delta y}{A \sinh \Delta y + (1 - A \cosh \Delta y) \tanh \xi \Delta y} \tag{72a}$$

or

$$t = \frac{e^{2\xi \Delta y} - 1}{e^{2\xi \Delta y}(1 - A e^{-\Delta y}) + A e^{\Delta y} - 1}. \tag{72b}$$

The most efficient form for the case $B < 1$ is Eq. (41) with z replaced by x throughout.

It is also instructive to study the general solution (41), and see how it reduces to different real representations for various ranges of A and B. This is carried out in [12]. It is easy to demonstrate that for $B < 1$, the solution is a scaled portion of a tangent, while for $B = 1$ it is a scaled portion of a rectangular hyperbola. For $B > 1$ (and the corresponding Δy given by Eq. (46)), the representation depends on the value of A. As shown in [12], if $\exp(-\Delta y) < A < \exp(\Delta y)$, the solution is a scaled portion of a hyperbolic tangent. Outside of that range the corresponding function is the hyperbolic cotangent, while on the boundary it is the exponential function.

When $A = 1$, the stretching function is antisymmetric, and the solution curve contains an inflection point. As A departs from one, the inflection point moves towards one end, and eventually could disappear. It is shown in [12] that an inflection point will be present if $1/\cosh \Delta y < A < \cosh \Delta y$ for $B > 1$, and $\cos \Delta x < A < 1/\cos \Delta x$ for $B < 1$. If $B < 2/\pi$, the solution must always contain an inflection point. The behavior of the solution in the B versus A plane, as well as the complex z plane, is illustrated in plots found in [12].

IV. A General Interior Stretching Function

In this section we derive a general interior stretching function $\xi(t, s_i, t_i)$, where s_i is the dimensionless slope at the inflection point t_i, i.e.,

$$s_i = \frac{d\xi}{dt}(t_i) \tag{73}$$

and

$$\frac{d^2\xi}{dt^2}(t_i) = 0 \qquad (0 \leqslant t_i \leqslant 1). \tag{74}$$

We limit our consideration to $s_i > 1$, which is the only case of practical interest. We again look for a function which is a scaled portion of an odd universal function $w(z)$. As in Section III, the simple functions $\sin z$ and $\tan z$ are considered first, and conditions (10) and (21) are examined for the antisymmetric case ($t_i = \frac{1}{2}$) when s_i is very large. The evaluation is carried out in [12], and $\sin z$ is found to produce an appropriate function with inverse length scales that are logarithmically of $O(1)$. The stretching function $\xi(t)$ is a scaled portion of the inverse hyperbolic sine.

The general interior stretching function for arbitrary t_i is readily obtained from the universal function $w = \sin z$, by letting $z = iy$. In terms of the range $\Delta y = y_1 - y_0$, and the implicity defined $\xi_i = \xi(t_i)$, the final result can be written as

$$t = t_i \left\{ 1 + \frac{\sinh[\Delta y(\xi - \xi_i)]}{\sinh \xi_i \Delta y} \right\}. \tag{75}$$

The inverse function is

$$\xi = \xi_i + \frac{1}{\Delta y} \sinh^{-1}[(t/t_i - 1) \sinh \xi_i \Delta y]. \tag{76}$$

The relation between ξ_i and t_i (for a given Δy) is obtained from Eq. (75) by setting $t = \xi = 1$. The result can be written as

$$1/t_i = 1 - \cosh \Delta y + \sinh \Delta y \coth \xi_i \Delta y. \tag{77}$$

Expressing the inverse hyperbolic cotangent in terms of a logarithm, we can write the inverse of Eq. (77) as

$$\xi_i = \frac{1}{2\Delta y} \log \left[\frac{1 + t_i(e^{\Delta y} - 1)}{1 - t_i(1 - e^{-\Delta y})} \right]. \tag{78}$$

Equations (76) and (78) are precisely the ones given by Thomas *et al.* [7].

If s_i and t_i are the given parameters, the corresponding value of Δy must be calculated. This can be done by differentiating Eq. (75) and substituting into Eq. (73). Using Eq. (77) to eliminate ξ_i, we can write the result as

$$\frac{1}{(s_i t_i \Delta y)^2} = \left[\frac{\cosh \Delta y - 1 + 1/t_i}{\sinh \Delta y} \right]^2 - 1. \tag{79}$$

This is an implicit equation for Δy involving two independent parameters. If the interior point is not too close to either end, and the slope s_i is sufficiently large, one can obtain a simplification. Assuming that $\exp(-2\Delta y) \ll 1$, we can approximate Eq. (79) as

$$2s_i\sqrt{t_i(1-t_i)} \simeq \frac{\sinh(\Delta y/2)}{\Delta y/2}. \tag{80}$$

Equation (80) is in the form of Eq. (62), and one can use the approximate analytic inversion given in Section III.

The special case of an antisymmetric solution is obtained by setting $t_i = \xi_i = \frac{1}{2}$. The results are

$$t = \frac{1}{2}\left\{1 + \frac{\sinh[\Delta y(\xi - \frac{1}{2})]}{\sinh(\Delta y/2)}\right\} \tag{81}$$

and

$$\xi = \frac{1}{2} + \frac{1}{\Delta y}\sinh^{-1}[(2t-1)\sinh(\Delta y/2)], \tag{82}$$

where

$$s_i = s_{1/2} = \frac{\sinh(\Delta y/2)}{\Delta y/2}. \tag{83}$$

A one-sided stretching function is obtained by setting $t_i = \xi_i = 0$. The results can be written as

$$t = \frac{\sinh(\xi\Delta y)}{\sinh \Delta y} \tag{84}$$

and

$$\xi = \frac{1}{\Delta y}\sinh^{-1}(t \sinh \Delta y), \tag{85}$$

where

$$s_i = s_0 = \frac{\sinh \Delta y}{\Delta y}. \tag{86}$$

It is interesting to compare the one-sided stretching function derived from the hyperbolic tangent (Eqs. (54)–(56)) with the function derived from the hyperbolic sine. Letting the subscripts T and S stand for the tangent and sine solutions, we see from Eqs. (54) and (86) that for a given s_0,

$$2\Delta y_{\mathrm{T}} = \Delta y_{\mathrm{S}}. \tag{87}$$

The maximum inverse length scale $L_{t\xi}^{-1}$ as defined by Eq. (2) occurs at $t = 0$ for the tangent solution, and has the value

$$((L_{t\xi}^{-1})_{\max})_{\mathrm{T}} = 2\Delta y_{\mathrm{T}} \tanh \Delta y_{\mathrm{T}}. \tag{88}$$

For the sine solution, the maximum occurs at $t = 1$, and has the value

$$((L_{t\xi}^{-1})_{\max})_{\mathrm{S}} = \Delta y_{\mathrm{S}} \tanh \Delta y_{\mathrm{S}}. \tag{89}$$

Thus, for s_0 large enough so that $\tanh \Delta y_{\mathrm{T}} \simeq \tanh \Delta y_{\mathrm{S}} \simeq 1$, the two solutions have the same maximum inverse length scale. The minimum slope $(d\xi/dt)_{\min}$ occurs at $t = 1$ for both solutions. The results are

$$((d\xi/dt)_{\min})_{\mathrm{T}} = \frac{\tanh \Delta y_{\mathrm{T}}}{\Delta y_{\mathrm{T}}} \tag{90}$$

and

$$((d\xi/dt)_{\min})_{\mathrm{S}} = \frac{\tanh \Delta y_{\mathrm{S}}}{\Delta y_{\mathrm{S}}}. \tag{91}$$

For large s_0, we thus obtain

$$((d\xi/dt)_{\min})_{\mathrm{S}} \simeq \tfrac{1}{2}((d\xi/dt)_{\min})_{\mathrm{T}}. \tag{92}$$

The one-sided function derived from the hyperbolic tangent thus has more points at the unclustered end ($t = 1$) than the one derived from the hyperbolic sine, for identical clustering at $t = 0$. The difference is because of the fact that the zero inflection point occurs at $t = 1$ for the first, and $t = 0$ for the second. The particular application would determine which of these two is preferable.

V. Concluding Remarks

In this work it has been assumed that the metrics and Jacobians that arise in the transformed equations are calculated by finite differences. If the equation $\mathbf{r}(t)$ of the ξ-coordinate curve is known analytically, and the transformation $t(\xi)$ is also given analytically, then the metrics and Jacobians can be analytically determined from the derivatives $d\mathbf{r}/dt$ and $dt/d\xi$. The truncation error in the numerical calculation will then be due to solely the finite-difference approximations to the derivatives of ϕ with respect to ξ. When ϕ varies monotonically with t, the optimum transformation would be one in which ξ varied linearly with ϕ, since this would result in zero truncation errors.

In order to compare transformations for the numerical and analytic treatment of Jacobians and metrics, consider a strictly one-dimensional case in which the single unknown v varies monotonically with distance x. Assume a highly localized interior region of rapid variation whose thickness is proportional to the small parameter ν. A simple example of such a solution is

$$v \sim \tanh(x/\nu), \tag{93}$$

where $x = 0$ in the localized region. This is actually the steady-state solution of Burgers' equation with fixed end conditions. For the analytic treatment of Jacobians and metrics, it follows that $\xi(x)$ should be a scaled portion of the hyperbolic tangent. The analysis of [2], which assumes a numerical treatment of Jacobians and metrics, shows that this choice is completely unsuitable, and instead favors a scaled portion of the inverse hyperbolic sine. If the differential equation is written so that only derivatives of v appear, then the hyperbolic tangent transformation should lead to a numerical solution with no truncation errors. But the equation can also be written in a form involving derivatives of several functions of v. An example is a strong conservation form. In this situation one has several variables $\phi(\xi)$ to be approximated by finite differences, and no single transformation $t(\xi)$ is optimum for all of them. If ν is very small, the hyperbolic tangent transformation would put all the interior grid points inside the region of rapid variation. There would be no points to resolve the boundary of this region. By contrast, the inverse hyperbolic sine transformation puts a sufficient number of points outside the region of rapid variation to resolve the complete one-dimensional region.

This discussion indicates that there are special situations and forms of the differential equations for which an analytic treatment of Jacobians and metrics can provide a desired accuracy with fewer grid points than the numerical treatment. These cases appear to be restricted to monotonic distributions that can be approximated by simple analytic expressions. For general applications of one-dimensional stretching functions, these special situations will not be met. It is then best to treat the Jacobians and metrics numerically, and use the stretching functions derived in Sections III and IV.

Another assumption in the derivation of the stretching functions is that the dimensionless length scale of the localized region of rapid variation could be extremely small. This requires the dimensionless slope of the transformation $d\xi/dt$ to be extremely large. If this condition is not encountered, and transformation slopes remain of $O(1)$, then the form of the stretching function is not critical. For example, many authors have used a scaled exponential as a one-sided stretching function. This is perfectly reasonable as long as the one-sided slope s_0 is not much larger than one. But one can readily show that the maximum inverse length scale is $\exp(s_0)$ for large s_0. Thus a simple exponential does not yield a suitable one-sided stretching function for very large slopes.

It should be made clear that Roberts was not the first one to use a stretching function involving the hyperbolic tangent. Mehta and Lavan [10], in an investigation of flow in a two-dimensional channel with a rectangular cavity, used a transformation based on the hyperbolic tangent to transform a semi-infinite region into a finite computational region, and to cluster grid points at the corners of the cavity. The maximum dimensionless slope, based on the length of the cavity, had a value of

0.8664 in their calculations. The transformation would have been a poor one if they had required a very large slope at the corner, as shown by the analysis of Section IV. The same authors [11] used a stretching function based on the inverse hyperbolic tangent in a study of the two-dimensional flow around an airfoil. A previous transformation had transformed the region external to the airfoil into a unit circle. The stretching function was necessary to cluster points further near the airfoil surface (to capture the boundary layer) as well as the free stream (to overcome the stretched grid produced by the first transformation). In their calculation, the nondimensional slopes at the two ends were 5.77 and 27.8. In this instance, the use of the hyperbolic tangent was both appropriate and necessary, as shown by the analysis of Section III.

An important criterion in the development of a two-sided stretching function is a continuous behavior as s_0 and s_1 varied from zero to infinity. This is necessary to obtain smooth grids constructed algebraically using one-dimensional stretching functions. For $B > 1$, the required function was found to be based on the inverse hyperbolic tangent. At first glance, the same function could be used for $B < 1$, simply by interchanging ξ and t in the expression. (This is what was actually done in the earlier stages of this work.) But this would violate the desired continuous behavior in the neighborhood of $B = 1$. The analytic continuation of the inverse hyperbolic tangent is the inverse tangent, which differs from the hyperbolic tangent. One can actually construct antisymmetric functions which are self invertible in this sense, but they do not include the elementary functions, and therefore would not be useful as stretching functions. Since the inverse tangent is not self invertible, it is necessary to use two different representations in calculating the stretching function numerically.

The use of the stretching functions derived in this work requires specifying their slopes at one or two points. The values are either obtained from matching slopes with another function, or estimating the length scale of a localized region where an appropriate dependent variable undergoes rapid variation. Admittedly, in a complex situation, the derivation of an appropriate length scale is not easy, and the value to assign to the slope can be somewhat arbitrary. Nevertheless, if a consistent criterion is used in assigning slope values, useful grids for numerical calculations can be generated. Recent examples of complex grids which were generated using the general two-sided stretching function are found in [13–18].

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